ASSET-LIABILITY MANAGEMENT FOR CZECH PENSION FUNDS USING STOCHASTIC PROGRAMMING

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Abstract

It is possible to model a wide range of portfolio management problems using stochastic programming. This approach requires the generation of input scenarios and probabilities, which represent the evolution of the return on investment, the stream of liabilities and other random phenomena of the problem and respect the no-arbitrage properties. The quality of the recommended capital allocation depends on the quality of the input scenarios and a validation of results is necessary. We propose scenario generation techniques and for output analysis in the context of defined contribution pension fund management. The application to the specific case of a Czech pension fund indicates the components that influence the recommended investment decisions and the fund's results. The initial position of the pension fund is important because of the accounting rules in the model and tracking both the market and purchasing valuation of assets.

Key words: Defined contribution plan, ALM, scenario-based stochastic programs, output analysis, case study

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1 ALM models for pension funds

There are many recent applications of ALM models with the main purpose—to support decisions of long-term investors who want to achieve certain goals and to meet future obligations. This concerns insurance companies, commercial banks, private investors, see e.g. [23, 30, 31, 32].

This paper is a contribution to ALM models for *pension funds*, which is the theme of the day for ageing populations of developed countries. There exist many types of pension plans and they have to respect various country-specific regulations.

A defined benefit plan (DBP) is linked to employment and it provides participants with a specified benefit during retirement. The retirement benefit is a fixed factor and benefit formulas are used to calculate it. They correspond to a combination of a flat amount, a flat percentage of earnings, a flat amount per year of service or a fixed percentage of earnings per year of service. Moreover, the benefits may be indexed to hedge against the inflation or may increase at a rate related with the increase of earnings of the fund. The fund manager is supposed to guarantee the benefit payments over a very long period of time, taking into account various country specific regulations concerning benefits/contributions levels, taxation and investment restrictions. She has to face various uncertainties such as the future life expectancy, inflation or return on investments. The contributions to the plan necessary to produce these defined benefits and the indexation policies are the variable factors to be decided.

We focus on *defined contribution plans* (DCP). Again, they may be linked to employment or profession and are mostly related to provisions taken out by individuals. Contributions accumulate on individual accounts of participants who participate in the profit sharing. At an agreed age, the pensions benefits are paid out as a lump sum or as a cash flow payments which are based on the accumulated wealth on the individual account of the participant and are calculated by actuarial techniques. According to the rules of the pension plan various forms of a less favorable terminal settlement may be paid to participants who want to leave the plan without being eligible for pension benefits. By regulations, pension funds are supposed to manage the accumulated contributions using *investment policies* which result in a stable growth of return. Besides of generating obligatory reserves, the profits are mostly shared among participants who are the main risk bearers if the fund defaults or the rules are changed. The contributions inflows and benefits outflows respect the pension plan rules, depend on demographic factors, legislative settings (for example the state support) and on specific behavior of individual participants. Besides of the dependence of benefits on the past pension plan performance, cf. the profit sharing, these cash flows may be treated as *independent* of important macroeconomic factors, such as returns on investments or inflation.

The performance of a pension fund may be analyzed by simulation, however, to support managerial decisions under uncertainty in the discrete time setting we rely on stochastic programming models. This approach is briefly summarized in the next Section; see also discussions in [6, 22, 31] and various applications presented in [31, 32] and in recent papers, e.g. [17, 18, 19, 21, 25].

An applicable ALM model for a Czech pension fund is developed in the third Section. The peculiarity of the asset/liability management for Czech pension plans is the lack of reliable historical asset return data. Moreover, the pension plans have not yet been fully stabilized partly due to the fact that during their relatively short history, managers of pension funds and participants of pension plans experienced several regulations changes. This puts limitations on the choice of scenario generation methods, see Subsection 2.1. Robustness of the results becomes a very important issue for the viability of the stochastic programming approach to the pension fund management and it is analyzed with the goal to detect the model inputs whose changes influence essentially the optimal investment policy. Applicable validation techniques are discussed in Subsection 2.2 and the relevant numerical results are given in Subsection 3.3.

2 Stochastic programs for pension funds management

For pension plans, both the future assets returns and liability streams of contributions and benefits are unknown. An application of stochastic programming means that uncertainties are modeled as random and that a discrete time model with a finite planning horizon is an acceptable choice. Another assumption which appears in theoretical formulations of stochastic programming models reads: *the probability distribution of random factors is known and independent of decisions*—a non realistic assumption in the context of pension funds uncertainties. The models are then applied with discrete probability distributions, carried by a finite number of atoms—*scenarios*; see [2, 15] for two-stage, multiperiod and multistage formulations of scenario-based stochastic programs.

The advantage of scenario-based stochastic programming models is their flexibility (e.g., the possibility to include decisions about investments, liabilities, various goals and various constraints, to reflect dynamic features) and their relative numerical tractability.

In financial applications of multistage stochastic programs, the generally accepted simplifying convention is that the portfolio can be rebalanced only at the beginning of certain periods (stages) to cover the goals. In the mean time, one applies a simplifying strategy, e.g., *buy-and-hold* or *fixed mix* allocations of returns, which does not assume any transactions except accumulating cash flows (coupons, dividends, etc.). Hence, to choose a suitable time discretization, stages and the horizon, cf. [10], is a strategic decision which should take into account the character of the problem in question, the existing information and various additional conflicting factors such as the quality of the approximation of the real decision process and the numerical tractability of the approach, which is also influenced by the available hardware and software.

The main interest lies with the first-stage decisions which consist of all decisions that have to be selected before the new information is revealed, just on the basis of the given (prescribed, known, approximated) probability distribution; in the context of ALM, the emphasis is on the initial asset allocation. The model should be solved repeatedly: after the first-stage decision is implemented and all parameters re-estimated taking into account new information, one applies the model with the rebalanced portfolio and with newly constructed scenarios (scenario trees) initiating from the actual values of the variables—*the rolling horizon approach.*

The objective function reflects the goals of the manager, e.g., to reach the best possible gains for the next year and at the same time to guarantee a long term prosperity in agreement with the regulations. The criterion is mostly related to the expected wealth at the end of the planning horizon. The risk factor can be incorporated into constraints, or it enters the objective function through a suitable utility function and penalty terms. Also criteria and constraints *nonlinear* in probability distributions can be applied; an example are models based on VaR.

The constraints follow the cash flow accounting rules and appear in the form of (time and scenario dependent) mostly *linear* constraints on cash and inventory balance and regulatory constraints. The guaranteed return constraint, cf. [6], the minimum funding requirement, cf. [4], or solvency requirements are often formulated as probabilistic constraints on the target value of the wealth, the funding level or the level of the accumulated wealth in relation to the total liabilities at the end of each period. Inclusion of probabilistic constraints, however, represents an increased complexity for numerical implementation of the model. For instance, in scenario-based models, binary variables are used to rewrite these (nonconvex) probability constraints; cf. [7]. Another possibility is to solve a sequence of suitably parametrized goal programming type models until the probability constraint is fulfilled, cf. [6, 16], or to incorporate the expected penalty due to various types of shortfalls into the objective function, e.g. [17, 31]. The last choice appears in our model.

For *scenario-based multistage stochastic programs* the input is usually in the form of a scenario tree. The nonanticipativity constraints on decisions may enter implicitly or in an explicit way. In both cases decisions based on the same history (i.e., on an identical part of several scenarios) are forced to be equal, as it is in the case of the first-stage decisions of the two-stage stochastic programs. With the explicit inclusion of the nonanticipativity constraints, the scenario-based multiperiod and multistage stochastic program with linear constraints can be written as a large-scale deterministic program

$$\max_{\mathcal{X}_0 \cap \mathcal{C}} \{ \sum_s p_s u^s(\boldsymbol{x}^s) \,|\, \boldsymbol{A}^s \boldsymbol{x}^s = \boldsymbol{b}^s,\ s = 1, \ldots, S \}. \tag{2.1}$$

Here $\mathcal{X}_0$ is a set of "hard" constraints, mostly simple constraints such as nonnegativity conditions, $\mathcal{C}$ is defined by the nonanticipativity constraints and u^s is the performance measure when scenario s occurs (with probability p_s).

The implicit inclusion of nonanticipativity constraints leads to the arborescent or nodal formulation of the stochastic program. Each node of the scenario tree corresponds to the history of the random process up to a certain time t, a stage at which decisions may be taken. The last decision point (stage) T corresponds to the chosen planning horizon τ which, depending on the model formulation, may be set as T or as $T+1$. Assuming discrete-time data processes the nodes may be numbered as $n = 1, \ldots, N$ with index $n = 1$ assigned to the root—the only node at stage $t = 1$. Nodes at stage t are indexed as (t, n) or taken as elements of the set $\mathcal{N}_t$ of nodes at stage t. The (unique) predecessor of node (t, n) at the stage $t-1$ is marked as $\hat{n}$. Let $\mathcal{D}(n)$ be the set of descendants of the node $n \in \mathcal{N}_t$; the elements of $\mathcal{D}(n)$ are then the nodes from $\mathcal{N}_{t+1}$ which can be reached from the node n. The probability of reaching the node (t, n) is p_{tn}. For planning horizon τ nodes belonging to the set $\mathcal{N}_\tau$ are called *leaves* and a scenario corresponds to a path from the root to some $n \in \mathcal{N}_\tau$. Given scenario probabilities $p_{\tau n}$ a path probability can be assigned to each node by a recursion.

At each node of the scenario tree (with exceptions of leaves) a decision $\boldsymbol{x}_n$ is taken. Constraints of (2.1) are rewritten as

$$\boldsymbol{W}_1\boldsymbol{x}_1 = \boldsymbol{b}_1,\ \boldsymbol{x}_1 \in \mathcal{X}_1,\ \boldsymbol{T}_n\boldsymbol{x}_{\hat{n}} + \boldsymbol{W}_n\boldsymbol{x}_n = \boldsymbol{b}_n,\ \boldsymbol{x}_n \in \mathcal{X}_n,\ n \in \mathcal{N}_t,\ t = 2, \dots, T \tag{2.2}$$

with matrices $\boldsymbol{W}_n$, $\boldsymbol{T}_n$ and vectors $\boldsymbol{b}_n$ resulting from the history preceding the node n. The set $\mathcal{X}_n$ is defined by separate constraints on $\boldsymbol{x}_n$. In this nodal formulation the objective function of (2.1) is

$$\sum_{n \in \mathcal{N}_\tau} p_{\tau n} u^n(\boldsymbol{x}_{\hat{n}}).$$

2.1 Scenario generation

To successfully apply stochastic programming models, one must design good input generation procedures, cf. [13], taking into account the existing information, software and hardware possibilities, and to develop suitable approaches for validation of results.

2.1.1 Scenario tree for assets

The scenario tree for assets is constructed independently of the scenario tree for liabilities due to the reasons discussed in Section 1. The selected procedure is related with the *choice of assets* or asset *classes* represented in our study by corporate and government bond indices and deposits. Moreover, due to the lack of historical data, the methods of scenario tree generation for assets have to adapt to a relatively low level of information. We apply the moment fitting method of [20] to create a scenario tree for returns of the considered assets clases.

The procedure is based on goal programming ideas where weighted squares of distances between the required values of moments of assets returns (e.g., mean, variance, skewness and kurtosis of the marginal probability distributions and both the in- and inter-stage correlations) and moments computed for the approximating discrete probability distribution are minimized. The proposed structure of the scenario tree, the required moments values and weights are the necessary input for the procedure, the output consists of optimally selected scenarios and probabilities.

Under Markov property of the assets returns, for example for the vector autoregressive processes (VAR) of the first order, the matching of moments can be run over collections of nodes in separate stages only:

Assume that the time discretization is chosen in agreement with the definition of stages and that the multidimensional time series distribution of returns r_{it} for assets $i = 1, \dots, I$ and $t = 1, \dots, T$, can be described by the VAR process

$$\boldsymbol{y}_t = \boldsymbol{\mu} + \boldsymbol{A}_1\boldsymbol{y}_{t-1} + \boldsymbol{e}_t,\ t = 1, \dots, T, \tag{2.3}$$

with $y_{it} = \ln(1 + r_{it})$, $i = 1, \dots, I, t = 1, \dots, T$.

Historical estimates are applied for matrix $\boldsymbol{A}_1$, for the second and higher order moments of the asset returns (including correlations), whereas for the first order moments, expert values may be accepted. The idea is that experts tend to take into account both the future expectations and the nonstationary behavior of the historical interest rates (caused partly by a specific policy of the Central Bank during the past currency crises). This makes the expert values more relevant then the historical averages.

An additional advantage of this scenario tree generation method is the possibility to test no-arbitrage property along the tree. Now we come to the final shape of the nonlinear optimization model for fitting the moments.

Let $\hat{n} \in \mathcal{N}_t$ be a node of the scenario tree at a given stage t and n, $n \in \mathcal{D}(\hat{n})$ its successor. The stage t is kept fixed and we omit now indices indicating the time period in question. The required model parameters $\sigma_{i,j}$, $i, j = 1, \ldots, I$, s_m, $k_m, m = 1, \ldots, I$, are (respectively) elements of the covariance matrix $\mathbf{\Sigma}$ of assets returns, their skewness and kurtosis (only for stages where more than 7 successors are considered), θ_i^{lo}, θ_i^{up} are bid and ask price of the asset i whose average price $P_{0i} = 1$, $i = 1, \ldots, I$, and e^{-r} denotes the one-period discount factor for the riskless asset. The weights consist of w_p—the weight for the no-arbitrage term, $w_{c,i,j}$—the weight for covariance for assets i, j, w_s, w_k—weights for skewness and kurtosis. For each node a different random vector of starting values u_i is used and it is transformed to $e_{\hat{n},n,i}$ using Choleski decomposition of $\mathbf{\Sigma}$.

The model variables are (real) $e_{\hat{n},n,i}$ disturbances, (positive) $p_{\hat{n},n}$ risk neutral arc probabilities and $\overline{p}_{\hat{n}}$ the minimum of risk neutral arc probabilities over successors of node $\hat{n}$.

Assuming that the average price of assets $P_{0i} = 1$, $\forall i$, the final condition for non-existence of arbitrage (cf. [24], Theorem 2) can be written as follows:

$$\theta_i^{lo} \leq e^{-r} * \sum_{n \in \mathcal{D}(\hat{n})} p_{\hat{n},n} * exp([\boldsymbol{\mu} + \boldsymbol{A}_1 \boldsymbol{y}_{t-1}]_i + e_{\hat{n},n,i}) \leq \theta_i^{up}, \forall \hat{n}, i = 1, \ldots, I, \tag{2.4}$$

with probabilities that have to fulfill conditions

$$\sum_{n \in \mathcal{D}(\hat{n})} p_{\hat{n},n} = 1, \, \overline{p}_{\hat{n}} \leq p_{\hat{n},n}, \forall n \in \mathcal{D}(\hat{n}). \tag{2.5}$$

In the goal programming objective function, deviations from the required moments values are penalized and positive risk neutral probabilities are rewarded at the same time. For *uniform* real probabilities we minimize at the considered stage t the function

$$\begin{aligned} \sum_{\hat{n} \in \mathcal{N}_t} \Bigg(-w_p * \overline{p}_{\hat{n}} + \sum_i \big(\sum_{n \in \mathcal{D}(\hat{n})} e_{\hat{n},n,i} / |\mathcal{D}(\hat{n})| \big)^2 + \\ \sum_{i,j} w_{c,i,j} \big(\sum_{n \in \mathcal{D}(\hat{n})} e_{\hat{n},n,i} * e_{\hat{n},n,j} / (|\mathcal{D}(\hat{n})| - 1) - \sigma_{i,j} \big)^2 + \\ w_s \sum_i \big((|\mathcal{D}(\hat{n})| - 1)^{(1/2)} \sum_{n \in \mathcal{D}(\hat{n})} e^3_{\hat{n},n,i} / (\sum_{n \in \mathcal{D}(\hat{n})} e^2_{\hat{n},n,i})^{(3/2)} - s_i \big)^2 + \\ w_k \sum_i \big((|\mathcal{D}(\hat{n})| - 1) \sum_{n \in \mathcal{D}(\hat{n})} e^4_{\hat{n},n,i} / (\sum_{n \in \mathcal{D}(\hat{n})} e^2_{\hat{n},n,i})^2 - k_i \big)^2 \Bigg) \end{aligned} \tag{2.6}$$

with respect to disturbances $e_{\hat{n},n,i}$, probabilities $p_{\hat{n},n}$ and $\overline{p}_{\hat{n}}$ for all i, $n \in \mathcal{D}(\hat{n})$ and $\hat{n} \in \mathcal{N}_t$ subject to (2.4) and (2.5). For numerical experiments and further caveats see [27].

2.1.2 Liability tree

The liability side of the ALM model of the defined contribution pension plan is driven by other factors, such as demographic data, legislative and plan regulations (retirement age, minimal required insured time). The economic factors essential for DBP, e.g. [3, 7], play a minor part in liabilities of DCP and the relatively low contributions of participants of Czech pension plans allow to neglect them when modeling liabilities in our study. Hence, the *liability tree will be generated independently of the evolution of various economic factors.*

A possibility is to *simulate the behavior of each participant,* described by a small number of attributes, such as age, sex, time spent in pension plan, quarterly contribution and type of pension. This provides a large number of observed instances of independent, equally distributed trajectories related with individual contracts. They are too many and do not form a scenario tree. Moreover, for our case study, just a sample of the individual contracts was available. It was used to estimate the corresponding two dimensional probability density of age and contribution level which serves as the basis for generation of a scenario tree of a desirable structure.

The marginal supports of the nonparametric estimate of the two dimensional probability density are discretized and the lattice points of the resulting two dimensional grid are interpreted as *representative participants* each of which passes through a finite number of states and corresponds to a certain number of individual contracts. The next step consists of

• detailed computations of flows of contributions, benefits and profit sharing settlement for representative participants;

• application of actuarial techniques and some heuristics to get transition probabilities (non-homogeneous Markov chain);

• simulation of paths—individual scenarios.

In spite of a crude approximation, aggregation and discretization technique, this simulation based approach described in [26] is understandable and enables to incorporate most of the details of the considered pension plan and legislative settings. The total net income F_t is obtained from simulated contributions and benefits of participants and the total profit sharing settlement λ_t is the sum of proportional *fixed* valorizations of the average annual levels of the individual accounts at stage t. The scenario tree of a given structure for (F_t, λ_t) is then constructed using a suitable method, such as the conditional sampling [5] applied in this study. The scenario tree is further reduced using the technique of [14] and the reduced liability tree is combined with the tree for assets.

2.2 Methods of output analysis

A natural question is *does the low level of information, the aggregation, simplifications and shocks cause essential errors?* To answer it, *robustness and sensitivity analysis* is necessary in the context of the applied ALM model. The short history does not allow us to apply historical backtesting. It is possible to compare results obtained with changed parameters, e.g., with alternative expert values of the first moments in the moment fitting technique, or to analyze the performance of the obtained investment decisions under out-of-sample or stress scenarios, etc. As indicated in this Subsection, such direct computational approaches may be complemented by error bounds; for general ideas see Chapter II.5 in [15] or [9]. We discuss the worst-case analysis with respect to liabilities and delineate the possible use of the contamination technique in stress testing and in analyzing the influence of including out-of-sample scenarios.

2.2.1 Worst-case analysis

Applicability of the method depends on specific assumptions concerning the structure of the problem and on the probability distribution. In the context of the ALM model for pension funds with an already fixed scenario tree for assets returns, *using only the expected liabilities* instead of random ones provides a (tight) *upper* bound on the fund performance; see [9] or [15], Chapter II.7, i.e., investment decisions based on the expected liabilities correspond to the most optimistic case. The question is if the uncertainty on the liability side of the problem can be neglected. A partial answer comes from the Value of Stochastic Solution (VSS), cf. [2], which quantifies the effect of using a nondegenerated probability distribution carried by multiple scenarios instead of the expected value scenario only. It is affected by the structure of the solved stochastic program, e.g., by the chosen planning horizon and by the scenario tree representing uncertainty, and it does not provide a generally valid answer.

2.2.2 Contamination technique and stress testing

Assume that the stochastic programming model for ALM such as (2.1) has been solved for a fixed set of scenarios ω^s, $s = 1, \ldots, S$, and that the influence of including other out-of-sample or stress scenarios should be considered. One could rewrite the program for the extended set of scenarios (and also constraints) and solve it. Another way is via the contamination technique, cf. [8, 9]. Rewrite the scenario-based stochastic program in the general form

$$\max_{\boldsymbol{x} \in \mathcal{X}} \sum_s p_s u^s(\boldsymbol{x}) \tag{2.7}$$

with a fixed set $\mathcal{X}$ of scenario-independent (first-stage) feasible solutions and with performance measures u dependent on scenarios.

Denote by P the probability distribution concentrated in ω^s, $s = 1, \ldots, S$ with probabilities $p_s > 0$, $\sum_s p_s = 1$, by $\varphi(P)$ the optimal value of the ALM model and assume that the set of optimal solutions of (2.7) is nonempty and bounded; let $\boldsymbol{x}^*(P)$ be one of optimal solutions. Inclusion of additional scenarios means to consider another discrete probability distribution, say Q, carried by the out-of-sample or stress scenarios indexed by $\sigma = 1, \ldots, S'$, with probabilities $q_\sigma > 0$, $\sum_\sigma q_\sigma = 1$ and to construct the *contaminated distribution*

$$P_\mu = (1-\mu)P + \mu Q \tag{2.8}$$

with a parameter $0 \le \mu \le 1$. The contaminated probability distribution is carried by the pooled sample of the $S + S'$ scenarios that occur with probabilities $(1-\mu)p_1, \ldots, (1-\mu)p_S, \mu q_1, \ldots, \mu q_{S'}$. It is possible to prove that the optimal value for the pooled sample $\varphi(P_\mu)$ is convex in μ and under mild assumptions, one gets a lower bound on its derivative at $\mu = 0$ as the difference between the value of the objective function $\sum_\sigma q_\sigma u^\sigma(\boldsymbol{x}^*(P))$ for the out-of-sample or stress scenarios evaluated at the optimal solution of the initial problem (2.7) and the initial optimal value. The bounds for the optimal value $\varphi(P_\mu)$ of the problem based on the pooled sample follow from convexity:

$$(1-\mu)\varphi(P) + \mu \sum_\sigma q_\sigma u^\sigma(\boldsymbol{x}^*(P)) \le \varphi(P_\mu) \le (1-\mu)\varphi(P) + \mu\varphi(Q) \tag{2.9}$$

for all $\mu \in [0, 1]$. If

$$\sum_\sigma q_\sigma u^\sigma(\boldsymbol{x}^*(P)) \ge \varphi(Q) - \varepsilon$$

then $\boldsymbol{x}^*(P)$ is an ε-optimal solution of

$$\max_{\boldsymbol{x} \in \mathcal{X}} \sum_\sigma q_\sigma u^\sigma(\boldsymbol{x}) \tag{2.10}$$

and the difference of the lower and upper bound in (2.9) is less or equal $\mu\varepsilon$. This quantifies the robustness of the results with respect to the out-of-sample or stress scenarios.

The additional numerical effort consists of

- Solving the problem (2.10) for the probability distribution Q carried by the out-of-sample, stress scenarios. The optimal decision is $\boldsymbol{x}^*(Q)$.

  In some papers *stress testing* is cut down to this procedure, i.e. to obtaining the optimal value $\varphi(Q)$ and comparing it with $\varphi(P)$. Such comparison, however, may be a cause of misleading conclusions: Assume, for example, that $\varphi(Q) = \varphi(P)$. With exception of the constant contaminated optimal value function $\varphi(P_\mu) = \varphi(P)\,\forall \mu \in [0, 1]$, the convexity arguments imply that there exist values of μ for which $\varphi(P_\mu) < \varphi(P)$.

- Evaluation and averaging the S' function values $u^\sigma(\boldsymbol{x}^*(P))$ for the new out-of-sample or stress scenarios at the already obtained optimal solution.

  This appears under the heading *stress testing* as well: one evaluates only the average performance of the obtained optimal solutions under the stress scenarios.

Similarly, one may view P_μ in (2.8) as the probability distribution Q contaminated by P (provided that the set of optimal solution of (2.10) is nonempty and bounded). The upper bound in (2.9) remains the same, the lower bound changes to

$$\mu\varphi(Q) + (1-\mu)\sum_s p_s u^s(\boldsymbol{x}^*(Q)) \le \varphi(P_\mu). \tag{2.11}$$

A joint exploitation of (2.9) and (2.11) provides a tighter lower bound valid again for all $\mu \in [0,1]$:

$$\max\{\mu\varphi(Q)+(1-\mu)\sum_s p_s u^s(\boldsymbol{x}^*(Q)),\ (1-\mu)\varphi(P)+\mu\sum_\sigma q_\sigma u^\sigma(\boldsymbol{x}^*(P))\} \leq \varphi(P_\mu). \quad (2.12)$$

These results may be exploited to *quantify* the changes of the obtained results when new, extremal circumstances are to be taken into account. This is a true robustness result.

Contamination bounds (2.9), (2.11), (2.12) are valid for all $0 \leq \mu \leq 1$. The weight μ may be interpreted as the degree of confidence in experts' view. Small values of μ are related to stability analysis, specific values of μ may provide equiprobable scenarios of the pooled sample.

The contamination technique can be useful not only in postoptimality analysis (inclusion of out-of-sample scenarios, emphasizing the importance of a scenario by increasing its probability) and stress testing but also in various stability studies, e.g., with respect to the assigned probabilities p_s. It is valid also for multistage problems and extends to integer stochastic programs. Technical details can be found in [8]; for an application see [11, 12], Chapter II.6 of [15] and Subsection 3.3.1.

3 Case study: ALM for a Czech pension fund

We now introduce a simple multistage stochastic programming model for asset-liability management of pension funds. Its formulation heavily depends on the legislative framework, on accounting methods and on current developments in the Czech Republic as described in the following subsection. At first, we give a more detailed description of the problem within the current state and legislative settings of supplementary pension insurance. We proceed then to model formulation. Finally, we present selected numerical results arising under diverse assumptions about economic and socio-demographic conditions, for various initial positions of the pension fund and for different risk attitudes of its manager. Using miscellaneous output analysis results we discuss stability and sensitivity properties of the model.

3.1 The problem and the input data

There are three important factors driving and restricting our modeling approach: the role of the supplementary pension insurance and legislative settings, the available data and the existing computer resources.

The *legislative framework* for supplementary pension insurance was given by [28] and has been in existence since 1994. Pension funds in the Czech Republic are private shareholder companies supervisored by the Ministry of Finance and strictly regulated in terms of their investments. Accumulated funds of pension funds may only be invested in government bonds, treasury bills, bonds issued by the Czech National Bank and other banks, mortgage certificates, corporate bonds and shares and participation certificates of unit trusts which are traded on the main and secondary market of the Prague Stock

Exchange, and bonds issued by OECD member states or by central banks of OECD member states. There are also limited possibilities to invest in real estate. The breakdowns of portfolios of pension funds (consolidated) show that more then 60% is invested in bonds, 25% in money deposits and treasury bills and less than 7% in shares and participation certificates, see for example [1]. Only defined contribution pension plans are allowed, except for the disability pension, where a defined benefit scheme appears, but its frequency is quite low.

Employers are allowed to contribute to pension funds for their employees and they enjoy tax deductions up to 3% of the gross wages. Similarly, employees do not have to tax contributions on pension insurance paid by the employer up to 5% of the gross wage. Both contributions are exempted from the base for the compulsory state social insurance (this saves 35% of each Koruna paid as contribution to the pension insurance instead of paying it as part of the gross wage). In addition, state contributions are added to contributions *paid* by participant. She/he is eligible for further tax deductions under additional circumstances. Tax deductions and state additional contributions settings are most advantageous for contributions up to roughly 12% of the average gross wage. Still, as a consequence of the tax regulations, the average contribution remains around 5% of the average gross wage only (year 2002).

Evidently, the state supplementary pension insurance is intended as a supplementary pillar and as such it is taken by the participants. Namely, accumulated assets per capita are just a little bit over 200 dollars (using the exchange rate of July 2002)-far behind the level in EU countries. In all pension funds, the number of participants receiving pension is less than 1% of all participants in the insurance portfolio and this is due to the short existence of the supplementary pension insurance and due to the prevailing requirement of participants to get their funds in the form of a lump sum compensation. More than 78% of participants are older than 40 years (year 1999).

The market saturation has reached a level comparable with developed countries and a harsh competition among pension funds has occured. Lower operating costs, multinational sound background and other financial services such as banking and mutual funds are the features which help the manager of the pension fund to become successful. Consolidation from the original 44 to 12 pension funds took place (year 2002) and it is expected to continue. The number of participants in the largest pension funds has reached more than $500,000$.

Attractiveness of the supplementary pension insurance is boosted by profits scored on accounts of participants, which for the years 2001 - 2002 ranged from 0 to 1 percent over inflation. Considering state contributions, this gives a nearly 9% valorization of accounts of participants for the same period. This is the key reason of the almost complete market saturation.

The model formulation, the choice of an appropriate method for generating scenarios and also the model validation techniques are affected by the *available data*. These constitute (in period 31.1.98-31.12.02) of monthly returns on indices of government bonds, high-rated corporate bonds and interest rates on deposits of sector of financial institutions. Three bond indices (total return weighted indices with the maturity of each instrument in the index longer than 1 year) are considered:

- corporate bond "blue chips", acronym $B1$,
- government bonds with weighted time to maturity equal 3 years (represents strategy in which the portfolio weights are adjusted in a prescribed specific way to preserve the required weighted term to maturity), acronym $B2$,
- government bonds with weighted term to maturity more than 3 years, acronym $B3$.

All securities appearing in indices are of rating A or higher. Hence, regarding the overall precision of the input data, the credit risk is taken as negligible.

Longer time series are not yet available which makes the historical backtesting impossible. In particular for government bonds, incurring of the state debt began in 1996 and as a liquid market instrument these established after 1997 only. Interest rates on deposits for financial institutions were taken as the base for returns on deposits in banks, no inflation adjusted interest rates are available. We did not have access to any privately constructed and computed indices of bond portfolios in the Czech Republic, Hungary or Poland.

The last restrictive assumption was made about *computer resources.* We aimed at implementing the model on PC running under Win 2000 1.2GHz AMD Duron, 750 MB memory with GAMS interface software accessing IBM OSL solvers.

Even though facing such a restrictive situation typical for transition countries with underdeveloped, thin financial markets and structures, the model should describe and quantify the most important uncertainties, in our case, *randomness of asset prices* and *random cash flows* due to contributions and benefits. It must also consider serious restrictions the manager faces: to respect the *initial conditions on portfolio composition*, to maintain *liquidity* to be able to meet demands on cash flows at given points in the future, and to consider the current *accounting practices* concerning additions to and release of financial provisions due to temporary fluctuation of asset prices and legislative settings for calculation of the accounting profit. For these reasons, both the past purchase prices and the market ones have to be distinguished and incorporated into the model. By recognizing the accounting standards our model differs from other stochastic programming ALM model formulations.

For numerically manageable stochastic programs the number of future decision points (i.e., of the stages) is limited to prevent an exponential growth of the event tree. Nevertheless, at given points in the future the results should support decisions how much and where to invest (investment classes, or portfolios), and to recommend an asset allocation under various assumptions about future development in a reasonable amount of time. The resulting size of the scenario tree depends on the *time discretization.* We work with the planning horizon of 3 years (i.e., with $T = 3$ and $\tau = 4$). A shorter horizon is not appropriate for asset allocation, which is a strategic decision. On the other hand, Czech pension funds are in a situation, where development of the population of insured is hard to predict and market shares of individual funds have not stabilized yet. Finally, the available data does not allow an extrapolation over a longer horizon. The decisions about portfolio rebalancing are taken in the time points when shareholders agree on the profit sharing, i.e., every end of the year.

To generate the scenario tree for assets over a planning horizon of several years, the method introduced by [20] was adapted, see Subsection 2.1.1, with one distinction: Because of the high nonstationarity of the available data, the VAR model (2.3) could not be fitted so the scenario tree for assets was constructed under the simplifying assumption of the interstage independence. Using monthly data, the moments of the probability distribution were recalculated to the yearly time step.

When modeling liabilities we dealt with quarterly contributions/benefits of participants and state contributions which is in agreement with the legislative settings. Later, these are aggregated to correspond with the yearly steps of rebalancing decisions.

Before proceeding to the model description we give the scheme of the whole machinery of ALM model inputs processing.

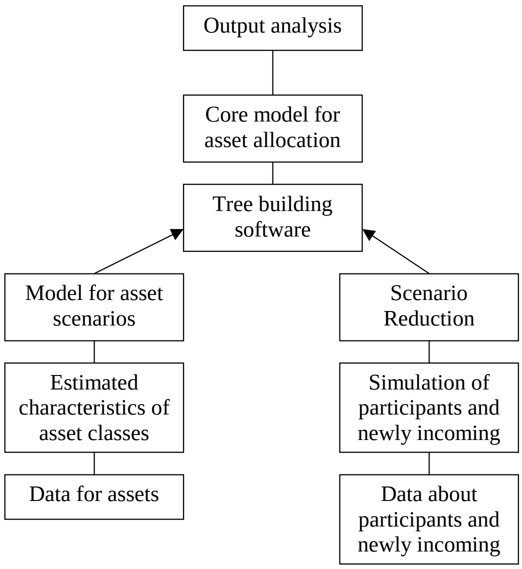

Figure 1: Structure of the Model

The whole procedure starts out to process the data inputs for the model. The left leg of Figure 1 is based on historical returns of the asset classes and other variables needed to describe the asset price dynamics. We proceed with estimation of the parameters of the probability distribution of assets returns. Following experts' judgements, different market situations may be considered. The estimated parameters enter the scenario generation procedure described in Subsection 2.1.1. The second leg of Figure 1 describes the development of the portfolio of the insured. The simulation-based approach delineated in Subsection 2.1.2 is applied.

When both the trees for scenarios of asset returns and scenarios of cash flows of contributions and benefits are completed the final scenario tree for the model is constructed. The model is solved and the output data proceed to the output analysis block. The whole machinery starts from the beginning unless outputs are satisfactory or all required alternatives have been analyzed.

The tasks are implemented using various software products trading the simplicity of implementation for its speed and automation, so efficiency is quite low. There are several time consuming steps of the procedure. The most demanding one is generation of the scenario tree for liabilities which lasts several hours, the next is generation of the scenario tree for assets which lasts approximately 90 minutes. The delivery of the model and its parameters to GAMS and the solution time needed to find an optimal solution of the ALM problem by OSL takes another 90 minutes.

3.2 The Model

Our stochastic programming model for asset liability management for Czech pension funds can be classified as a scenario-based stochastic program with linear constraints. Some features related to accounting practices might require integer variables. Still, we relax the integrality requirements and present here a simplified version of reality to keep the model numerically tractable. The objective is to maximize the expected terminal wealth minus the expected total penalization of shortfalls over stages. The penalty function is downside quadratic and accounts for not reaching the required (predetermined) valorizations. The model records a detailed information about the initial buying price of the assets classes, their market price is tracked as well. We have chosen this approach to be able to model some requirements on financial provisions and to be able to *distinguish between accounting profit* and *cash flow*, which is crucial for pension fund manager working under Czech legislative settings.

We will keep the notation introduced in Section 2. The nodal form (2.2) is used when describing the constrains but we prefer the form (2.1) when delivering the model to the GAMS solver. First, model parameters, coefficients and variables will be listed. After that, we explain the model constrains and its objective function in detail.

MODEL DICTIONARY
model parameters:

r risk free rate, used for capitalization of profits in the objective function,

α the coefficient for addition to financial provisions,

W_1 the wealth (in market value) of the pension fund at the beginning of the period $(1, 2)$,

β_i the coefficients for transaction costs, expressed as percentages of value sold or bought, $i = 2, \dots, I$,

γ the weight given to the penalty term in the objective.

For period $(t, t+1)$, $n \in \mathcal{N}_{t+1}$,

d_{t+1} the discount factor, $d_{t+1} = (1+r)^t$ with r the one period risk free rate,

$r_{1,t+1,n}$ the rate of return on deposits,

$r_{i,\hat{t},t+1,n}$ the rate of return on bond index i bought at the beginning of period $(\hat{t},\hat{t}+1)$, held at the beginning of the period $(t+1,t+2)$,

$F_{t+1,n}$ the aggregated cash flows from the participants,

$\lambda_{t+1,n}$ the sum of proportional *fixed* valorizations of the average annual levels of the individual accounts,

$p_{t+1,n}$ the probability of scenario leading to node n;

decision variables:
(nonnegative) $n \in \mathcal{N}_{t+1}$, $i = 2,\dots,I$,

$X^h_{i,\hat{t},t+1,n}$ the holdings of bond index i bought at the beginning of period $(\hat{t},\hat{t}+1)$, held at the beginning of the period $(t+1,t+2)$, after rebalancing, valued in purchase prices average (money stake),

$X^s_{i,\hat{t},t+1,n}$ the amount of bond index i bought at the beginning of period $(\hat{t},\hat{t}+1))$, sold at the beginning of the period $(t+1,t+2)$, valued in purchase prices average (money stake),

$X^b_{i,t+1,n}$ the amount of bond index i bought at the beginning of period $(t+1,t+2)$, valued in purchase prices average (money stake);

other variables:
(nonnegative) at the beginning of the period $(t+1,t+2)$, $n \in \mathcal{N}_{t+1}$,

$X_{1,t+1,n}$ deposits,

$Y_{t+1,n}$ financial provisions,

(real) for period $(t,t+1), n \in \mathcal{N}_{t+1}$,

$C_{t+1,n}$ additions/release to financial provisions,

$\pi_{t+1,n}$ accounting profit/loss.

Asset Inventory Equation. The asset inventory equation differs for the first period after buying the asset $i \in \{2,\dots,I\}$ and for the next periods. No cash flows from these assets (bond indices) arise.

$$X^h_{i,h,t-1,t,n} = X^b_{i,t-1,\hat{n}} - X^s_{i,t-1,t,n}, n \in \mathcal{N}_t, t = 1,\dots,T, \tag{3.1}$$

$$X^h_{i,h,\hat{t},t,n} = X^h_{i,\hat{t},t-1,\hat{n}} - X^s_{i,\hat{t},t,n}, \hat{t} = 0,\dots,t-2, n \in \mathcal{N}_t, t = 2,\dots,T.$$

The inventory equation for the deposit account, $i = 1$, will be specified in the cash balance equation (3.4).

Additions to Financial Provisions and Financial Provisions Accumulation. Additions to provisions are often demanded by an auditor to assure that today's profit shared among participants of the pension plan does not reduce the opportunity to attain similar profits in the following periods. This reasoning is enforced only if the current portfolio might suffer losses due to the price decline on the market. The mentioned feature is modeled by the requirement on including a part of the *experienced* capital losses in the computation of the accounting profit, similarly as in practice. This is implemented by additions to provisions for the riskier assets $i \in \{2, \ldots, I\}$.

$$Y_{t+1,n} \geq -\alpha\Big(\sum_{i=2}^{I}\big(\sum_{\hat{t}=0}^{t-1} r_{i,\hat{t},t+1,n} X^h_{i,\hat{t},t,\hat{n}} + r_{i,t,t+1,n} X^b_{i,t,\hat{n}}\big)\Big), \tag{3.2}$$

$$C_{t+1,n} = Y_{t+1,n} - Y_{t,\hat{n}},\ t = 1, \ldots, T, n \in \mathcal{N}_{t+1}.$$

Equation (3.2) specifies additions or release of financial provisions in case of realized capital losses for the current period. If necessary, financial provisions are added and they might be also released, but still kept on the minimal required level given by α.

Profit and Loss Accounting. Accounting profit/loss calculation involves proceeds from sales of assets minus the purchase price of assets sold (the first term in (3.3)), minus transaction costs expressed as the percentage of proceeds and expenses (the second term in (3.3)), minus additions to financial provisions (or plus release of provisions as $C_{t+1,n}$ might be positive or negative), plus the return on the deposit account—the last term in (3.3). The return on the deposit account may be expressed in a more detailed way, see (3.4).

$$\pi_{t+1,n} = \sum_{i=2}^{I}\sum_{\hat{t}=0}^{t-1} r_{i,\hat{t},t,\hat{n}} X^s_{i,\hat{t},t,\hat{n}} - \sum_{i=2}^{I} \beta_i\Big(\sum_{\hat{t}=0}^{t-1}(1 + r_{i,\hat{t},t,\hat{n}}) X^s_{i,\hat{t},t,\hat{n}} + X^b_{i,t,\hat{n}}\Big) \tag{3.3}$$

$$-C_{t+1,n} + \frac{r_{1,t+1,n}}{1 + r_{1,t+1,n}}(X_{1,t+1,n} - F_{t+1,n}),\ t = 1, \ldots, T,\ n \in \mathcal{N}_{t+1}.$$

Taxes are not included as pension funds enjoy a special tax regulation which makes taxation negligible.

Cash Balance Equation. The cash balance equation specifies that all the money on the deposit account at the beginning of the period plus cash inflow (asset selling, interest on deposit account) minus cash outflow (asset buying, transaction costs) plus cash flow related to liabilities (contributions and benefits) must equal the amount of money on the deposit account at the end of the period.

$$X_{1,t+1,n} = \big(X_{1,t,\hat{n}} + \sum_{i=2}^{I}\sum_{\hat{t}=0}^{t-1}(1 + r_{i,\hat{t},t,\hat{n}}) X^s_{i,\hat{t},t,\hat{n}} - \sum_{i=2}^{I} X^b_{i,t,\hat{n}} \tag{3.4}$$

$$-\sum_{i=2}^{I} \beta_i\Big(\sum_{\hat{t}=0}^{t-1}(1 + r_{i,\hat{t},t,\hat{n}}) X^s_{i,\hat{t},t,\hat{n}} + X^b_{i,t,\hat{n}}\Big)\big)(1 + r_{1,t+1,n}) + F_{t+1,n},$$

$$t = 1, \ldots, T, n \in \mathcal{N}_{t+1}.$$

Penalization of lower than promised Accounting Profit. The fixed valorization of the accumulated wealth is used when simulating scenarios for liabilities, see Subsection 2.1.2. This helps to avoid the dependency of the input probability distribution on decision variables. An unfavorable situation arises if $\lambda_{t+1,n} - \pi_{t+1,n}$, the difference between the sum of the proportional *fixed* valorizations of the average annual levels of the individual accounts and the computed accounting profit, is positive. We rewrite it as the difference of two *positive* slack variables $M^{pr,-}_{t+1,n}$, $M^{pr,+}_{t+1,n}$:

$$\lambda_{t+1,n} - \pi_{t+1,n} = M^{pr,-}_{t+1,n} - M^{pr,+}_{t+1,n},\ t = 1, \ldots, T, n \in \mathcal{N}_{t+1}. \tag{3.5}$$

Positive values of $M^{pr,-}_{t+1,n}$, rescaled to be commensurable with the main term (wealth at the planning horizon) in the objective function, are penalized using the downside quadratic penalty function which is subsequently approximated by a piece-wise linear function in a standard way: Given a partition δ_j, $j = 0, \ldots, J$, with $\delta_0 = 0$

$$M^{pr,-}_{t+1,n}/r = \sum_{j=1}^{J} M_{j,t+1,n},\ t = 1, \ldots, T, n \in \mathcal{N}_{t+1}, \tag{3.6}$$

where

$$M_{j,t+1,n} \leq \delta_j - \delta_{j-1}, j = 1, \ldots, J.$$

Objective of the ALM Model. The objective reflects the goals of the pension fund manager. On the one hand she is forced to reach the highest gains for the next year so the annual rate of return on funds of the participants is the highest possible, on the other hand she cannot afford to sell out assets promising outstanding returns in the future. Moreover she should control prospective capital losses recorded up to the planning horizon $\tau = T + 1$ otherwise she will not meet the standards set via $\lambda_{t+1,n}$ and she will be exposed to additions to financial provisions. She must also maintain the liquidity of the pension fund. From the long term perspective the core problem is the growth of the value of funds.

These ideas naturally lead to maximization of the expected wealth at the planning horizon $\tau = T + 1$ discounted to the beginning of the planning horizon minus the discounted expected penalty for the shortfalls $(\lambda_{t+1,n} - \pi_{t+1,n})^+$.

$$\begin{aligned} d_{T+1} * \sum_{n\in\mathcal{N}_{T+1}} p_{T+1,n} \Big(X^h_{1,T+1,n} + \sum_{i=2}^{I} \big(\sum_{\hat{t}=0}^{T-1} (1 + r_{i,\hat{t},T+1,n}) X^h_{i,\hat{t},T,\hat{n}} + (1 + r_{i,T,T+1,n}) X^b_{i,T,\hat{n}}\big) \Big) \\ - \gamma * \sum_{t=1}^{T} d_{t+1} \sum_{n\in\mathcal{N}_{t+1}} p_{t+1,n} \sum_{j=1}^{J} \eta_j * M_{j,t+1,n}. \end{aligned} \tag{3.7}$$

Here, η_j, $j = 1, \ldots, J$, are the slopes in the piecewise linear approximation of the downside quadratic penalty function valid on the interval $[\delta_{j-1}, \delta_j]$, $j = 1, \ldots, J$. The parameter γ reflects the degree of the risk aversion of the fund manager and for purposes of the output analysis it is rescaled as

$$\gamma = \frac{a}{W_1} \tag{3.8}$$

with a related to the manager's risk aversion.

Validity of the asset accumulation and other equations was checked via balance sheets and income statements constructed in each node of the scenario tree.

3.3 Selected numerical results and output analysis

Using outputs of a model without any further validation may lead to serious problems as the obtained optimal solution may perform very poorly under a different input specification. The aim of this Subsection is to evaluate the model behavior under various assumptions about economic and demographic scenarios and to test its sensitivity on selected input parameters, such as the weight of the penalty term and the initial balance sheet.

In the first part of the numerical study, Subsection 3.3.1, we provide the contamination bounds for the optimal value of the objective function when additional, out-of-sample or stress scenarios are included. Another, frequently used method for validation of results is the historical backtesting based on historical time series. It was impossible to apply it as the historical time series are still too short and nonstationary.

Our model for defined contribution plan was built under specific assumptions discussed in Subsection 3.1 and it reflects the legislative regulations and accounting rules used in the Czech Republic. Namely, an adequate inclusion of creation and release of provisions requires a detailed tracking of assets prices, remembering for each asset both its historical cost and the current market price. Moreover, distinction between cash flows and accounting categories requires introduction of F_t and λ_t.

Such a detailed treatment of assets gives us a chance to inspect changes of the optimal portfolio for different variants of the initial balance sheet. All these variants have the same asset weights in the initial portfolio composition when using market values, but different asset weights when using historical costs. These variants will also differ in the level of provisions. The goal is to show that provisions creation or release is a very influential factor which cannot be omitted; see Subsection 3.3.2.

In the third part of this numerical study we try to answer the question to what extent is the optimal portfolio and the objective value influenced by incorporating random liabilities and by changes in scenarios for liabilities. The analysis based on the Value of Stochastic Solution (VSS, cf. [2]) does not provide a full answer to the first question, as it is based only on comparison of results for five liabilities scenarios obtained via the scenario reduction algorithm with results for the expected value scenario. It was not tested for a higher number of scenarios for liabilities as it would increase too much the demands on the available computer resources.

The second question concerns sensitivity of the optimal portfolio composition to changes in behavior of participants. As a stress situation, decline of newly incoming and higher propensity towards the lump sum compensation will be considered.

The available data were described in Subsection 3.1. Here we briefly summarize that these are monthly data (monthly returns), which were annualized before estimating correlations and other statistical parameters of returns. These time series were divided into periods where stationarity was assumed. This was done using experts' views about data and events which might have disrupted stationarity and using graphs about development of the indices in question. As mentioned in 3.1, we were not able to fit the VAR model to these time series because of their shortness and a kind of a “trending” tendency, possibly caused by the aftermath of the monetary crisis in 1997. In 1999–2001 the interest rates displayed a decreasing tendency as the Czech National Bank was lowering the key interest rate (similarly as most Central Banks world wide). Mean values for the planning horizon τ (recall that τ covers three one year periods) were set roughly in correspondence with returns experienced in the the year 2002.

Before proceeding to the output analysis we describe the selected variants, introduce their acronyms and list the input data that will be used later on.

VARIANTS AND ACRONYMS

R rally of the market; statistical parameters of assets returns (mean values, covariances, skewnesses, kurtosises) estimated from "historical" data and the scenario tree of the structure $(20, 8, 5)$ (here, 8 stands for 8 successors of each node in the first period, etc.) is constructed;

S slump of the market; statistical parameters as covariances, skewnesses, kurtosises estimated again from the corresponding "historical" data and the scenario tree is constructed to have the structure $(10, 8, 8)$. The mean values for asset returns were set to 40% of mean values for variant R;

$Dep, B1, B2, B3$ denotes deposits and indices described in Subsection 3.1,

$CL1, CL2, CL3$ variants assume equal capital losses on the value of portfolio in the initial balance sheet but different levels of provisions. The initial market values of the portfolio assets wb are equal and are the same as in variant NL;

NL no capital loss is assumed for any asset in the initial balance sheet.

The initial conditions on asset proportions and their valuation are summarized in the balance sheet; the variant NL is given below.

Dep		8.72	AR	0.01
$B1$	MV	6.47	G	22.65
$B1$	HC	6.47	RE	1.06
$B2$	MV	6.47	Y	0.00
$B2$	HC	6.47		
$B3$	MV	2.06		
$B3$	HC	2.06		
$Total$		23.73	$Total$	23.73

Table 1: Balance sheet at the beginning of the planning horizon, *variant* NL

The symbols used in the balance sheet:

MV, HC market and historical cost (purchase price) valuation,

AR asset revaluation,

RE retained profits,

G other capital funds (accounting item for funds contributed by participants of the pension plan),

Y financial provisions.

In the variant NL, the market and historical values are equal and no capital loss is considered.

The model uses Czech Koruna for currency, rescaled to $1e08$ units. Statistical inputs to the asset scenario generating model (2.6) are summarized in Tables 2 and 3 for variants R and S, respectively.

	mean	*variance*	*skewness*	*kurtosis*	*correlations*			
					Dep	$B1$	$B2$	$B3$
Dep	0.042	2.23e-6	-0.06	1.85	1			
$B1$	0.074	6.85e-4	-0.45	3.06	0.057	1		
$B2$	0.079	5.70e-4	-0.09	2.08	0.053	0.901	1	
$B3$	0.102	1.71e-3	-0.14	2.20	-0.056	0.883	0.944	1

Table 2: Parameters of asset return distribution for variant R

	mean	*variance*	*skewness*	*kurtosis*	*correlations*			
				Dep	$B1$	$B2$	$B3$	
Dep	0.017	2.05e-6	0.95	3.30	1			
$B1$	0.030	8.84e-4	-0.42	2.16	-0.132	1		
$B2$	0.031	7.89e-4	0.33	1.87	-0.232	0.949	1	
$B3$	0.041	2.20e-3	0.48	1.85	-0.182	0.906	0.953	1

Table 3: Parameters of asset return distribution for variant S

The downside quadratic penalty function is approximated by $J = 5$ linear segments whose parameters are

δ_1	δ_2	δ_3	δ_4	δ_5
0.5	1	2	4	∞
η_1	η_2	η_3	η_4	η_5
0.5	1.5	3	6	12

Table 4: Parameters of the approximated penalty function

Table 5 describes the reduced scenario tree for liabilities obtained by the procedure explained in Subsection 2.1.2 which is used in computations unless stated otherwise, and the expected value “tree” consisting of one scenario only.

Scenario	*Period*			*Probability*
	F			
	1	2	3	
1	2.283e8	3.484e8	1.273e8	0.189
2	2.291e8	2.728e8	1.242e8	0.182
3	2.291e8	2.728e8	6.494e7	0.170
4	2.003e8	2.963e8	8.381e7	0.193
5	2.153e8	3.362e8	9.007e7	0.266
	λ			
1	7.182e7	9.326e7	1.025e8	
2	7.174e7	9.093e7	9.981e7	
3	7.174e7	9.093e7	9.788e7	
4	7.084e7	9.047e7	9.811e7	
5	7.128e7	9.230e7	1.003e8	
	EF			
1	2.197e8	3.085e8	9.784e7	1
	$\sigma(F)/EF$			
	0.051	0.102	0.235	
	$E\lambda$			
1	7.146e7	9.164e7	9.981e7	1
	$\sigma(\lambda)/E\lambda$			
	0.005	0.011	0.016	

Table 5: Scenario tree for liabilities, after reduction

The model formulation involves further *parameters that are fixed for all variants*: The risk free interest rate $r = 3$ as assumed in actuarial computations for pension plans, the coefficient α in (3.2) equals 0.1, coefficients for transaction costs $\beta_i = 0.01\,\forall i$ and the initial market value of the portfolio $W_1 = 23.73$. These parameters can be easily calibrated as they have a clear economic interpretation. A bit unclear is setting of the parameter γ, which assigns the weight to the penalty term in the objective function (3.7). Its value describes the manager's attitude towards the situation when, under given scenario tree for stochastic parameters, the decisions do not provide at least the required fixed valorizations of the personal accounts of the participants. In our case, fixed valorizations $3.25, 3.5, 3.5\%$ p.a. are assumed for the three year planning horizon respectively.

To set a value of γ for our numerical experiments we inspect first the change of the optimal portfolio in variant R/NL with the standard setting of liability scenarios (see Table 4) for different choices of a in (3.8). The results (expected values of the portfolio composition) are in the Table 6.

a	Dep	B1	B2	B3	Dep	B1	B2	B3	Dep	B1	B2	B3
	end of first period (%)				end of second period (%)				end of third period (%)			
0.04	0	0	25	75	0	0	0	100	0	0	0	100
0.4	64	2	26	8	38	0	0	62	10	0	0	90
1	78	0	14	8	57	0	0	43	17	0	0	83
2	84	0	8	8	66	0	0	34	23	0	0	77
4	90	0	1	9	74	0	0	26	31	0	0	69
40	97	0	0	3	86	0	0	14	45	0	0	55

Table 6: Portfolio composition over periods

The acceptable parameter values are $a = 0.4$, $a = 1$, $a = 2$ and $a = 4$ which do not lead to corner solutions. In our numerical experiments, we use the value $a = 4$ which corresponds to a moderately conservative investment style typical for many pension funds, cf. [31], and $a = 0.4$ representing the low propensity to risk aversion of the fund manager.

3.3.1 Contamination technique

We now evaluate the impact of including additional "out-of-sample" scenarios on the optimal value of the objective function (3.7), using the contamination technique explained in Subsection 2.2.2. We assume that variant R is the base variant (probability distribution P) and variant S is the variant representing "out-of-sample" or stress scenarios (probability distribution Q). For both variants, the initial conditions on asset proportions and their valuation are equal, see the balance sheet in Table 1, and the scenario tree for liabilities is fixed according to Table 5.

Separately for each variant, the optimal portfolio composition for the first period and the expected development of the wealth of the pension fund over the subsequent periods is given below.

a	Market	Portfolio				Median of wealth of PF		
		(after rebalancing, market values %)				(% of initial wealth), end of period		
		Dep	B1	B2	B3	1	2	3
0.4	R	62	2	27	9	115	136	154
	S	37	27	27	9	112	127	133
4	R	90	0	1	9	114	133	148
	S	37	27	27	9	112	127	133
		Portfolio (initial, market values %)				Total	Sum	
		37	27	27	9	100	23.73	

Table 7: Portfolio composition

Regardless of the value of a, it is optimal under variant S to keep the same portfolio weights as in the initial balance sheet Table 1. Inspecting the expected portfolio composition in later stages shows that a gradual shift toward cash, up to ninety percent of the expected weight in the last period is optimal. On the contrary, under variant R, where assets are assumed to have a higher expected value, it is optimal to sell $B1$, $B2$ and keep $B3$—the asset with the highest expected return. Considering the high positive correlation of $B1, B2, B3$ this behavior might be expected. In later periods the expected optimal portfolio weights shift towards a deeper position in $B3$. The magnitude of the shift depends on the value of a.

Figure 2 demonstrates the contamination bounds obtained according to (2.12) for $a = 0.4$ and $a = 4$, value $\mu = 1$ corresponds to variant S. For $a = 0.4$, the bounds for the optimal value of (3.7) for the pooled sample $R\&S$ with weights μ and $1 - \mu$, respectively, are very narrow over the whole interval $[0, 1]$. The wish to have equiprobable scenarios of the pooled sample means to use $\mu = 5/9$. The contamination bounds provide an interval in which the optimal value $\varphi(P_\mu)$ for the pooled sample is contained, i.e. $[-7.55, -2.5]$ for $a = 4$. The directional derivative of the optimal value function at $\mu = 0^+$ both for $a = 0.4$ and $a = 4$ is negative, hence, as expected, the optimal value does not increase when including the stress scenarios S regardless of the weight $1 - \mu$, $\mu \in [0, 1]$.

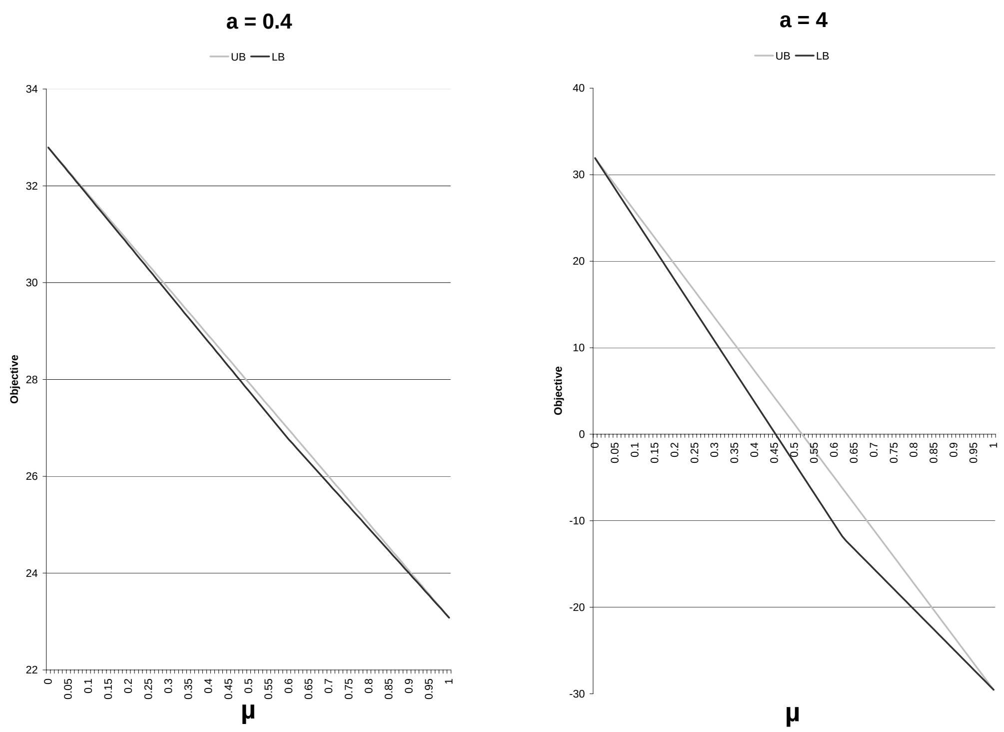

Figure 2: Objective value bounds for the pooled sample of scenarios $R\&S$

The fund manager is interested in robustness of the attained expected terminal wealth, see the first term in the objective function (3.7). For the optimal investment policy $\boldsymbol{X}^*(P)$ obtained by solving (3.1)–(3.7) with $a = 4$, the expected discounted terminal wealth, $W_{T+1}(\boldsymbol{X}^*(P), P) = 32.12$. equals 32.12. If the stress variant S occurs instead of R, it changes to $W_{T+1}(\boldsymbol{X}^*(P), Q) = 28.66$ For the pooled sample $R\&S$

$$W_{T+1}(\boldsymbol{X}^*(P), P_\mu) = (1 - \mu) W_{T+1}(\boldsymbol{X}^*(P), P) + \mu W_{T+1}(\boldsymbol{X}^*(P), Q),$$

is a linear function of μ. Hence, the expected terminal wealth for the pooled $R\&S$ sample with the contamination weight $\mu = 5/9$ equals 30.2.

3.3.2 Dependence of the optimal portfolio on the initial balance sheet

The level of provisions versus the historical costs of assets and market values of assets are categories tied by accounting practices. For example: the selling of asset at the beginning of the first period in which the market price is lower than the historical price means a decrease of the profit (increase of the loss) and at the same time an increase in the profit (or decrease of the loss) due to release of provisions established to cover this loss. If the provisions were not set in previous periods on a sufficiently high level (provisions are agreed on with an auditor, forecasting of the price development is subject to an instantaneous change) then selling of the asset will influence accounting profit, hence the penalization in our model and the optimal solution as well.

This indicates that the optimal portfolio depends not only on the asset proportions when assets are valued in market prices but also on historical costs of the assets and on the level of provisions. We inspect this dependence using balance sheets having equal proportions of assets valued in market prices but different historical costs. The balance sheets are in Tables 8–10; compare with Table 1.

Dep		8.72	AR	-2.93
$B1$	MV	6.47	G	22.65
$B1$	HC	6.47	RE	3.40
$B2$	MV	6.47	Y	0.60
$B2$	HC	6.47		
$B3$	MV	2.06		
$B3$	HC	5.00		
$Total$		23.73	$Total$	23.73

Table 8: Balance sheet *variant* $CL1$

Dep		8.72	AR	-2.93
$B1$	MV	6.47	G	22.65
$B1$	HC	6.47	RE	3.85
$B2$	MV	6.47	Y	0.15
$B2$	HC	6.47		
$B3$	MV	2.06		
$B3$	HC	5.00		
$Total$		23.73	$Total$	23.73

Table 9: Balance sheet, *variant* $CL2$

Dep		8.72	AR	-2.93
$B1$	MV	6.47	G	22.65
$B1$	HC	6.47	RE	3.71
$B2$	MV	6.47	Y	0.29
$B2$	HC	6.47		
$B3$	MV	2.06		
$B3$	HC	5.00		
$Total$		23.73	$Total$	23.73

Table 10: Balance sheet, *variant* $CL3$

The only difference among these balance sheets is in the level of provisions. Reminding the meaning of the parameter α in (3.2) we can compare these variants using the ratio $\rho := \frac{Y}{|AR|}$. We have $\rho = 0.2$ for $CL1$, $\rho = 0.05$ for $CL2$ and $\rho = 0.1$ for $CL3$. Table 11 summarizes the results.

a	Balsheet	Market	Portfolio				Median of wealth of PF		
			(after rebalancing, market values %)				(% of initial wealth), end of period		
			Dep	B1	B2	B3	1	2	3
0.4	CL1	R	22	28	27	23	116	140	159
		S	31	27	27	15	112	127	134
	CL2	R	61	3	27	9	115	136	154
		S	32	27	27	14	112	127	134
	CL3	R	48	16	27	9	115	138	156
		S	32	27	27	14	112	127	134
	NL	R	62	2	27	9	115	136	154
		S	37	27	27	9	112	127	133
4	CL1	R	33	27	27	13	116	139	158
		S	34	27	27	12	112	127	134
	CL2	R	91	0	0	9	114	132	146
		S	35	27	27	11	112	127	134
	CL3	R	78	0	13	9	114	134	149
		S	35	27	27	11	112	127	134
	NL	R	90	0	1	9	114	133	148
		S	37	27	27	9	112	127	133
			Portfolio (initial, market values %)				Total	Sum	
			37	27	27	9	100	23.73	

Table 11: Portfolios at the beginning of the first stage after rebalancing

Table 11 shows that the optimal portfolios for the first period of the model are *different* even though the initial portfolio weights calculated using market values are identical. It means that the historical costs of portfolio assets and the initial level of provisions do matter. The higher the initial level of provisions (higher ρ) the higher are the weights in the optimal portfolio for $B1, B2, B3$. Particularly the weight of $B3$, the asset with the highest volatility, increases with an increase of ρ. Hence, provisions are an important factor that influences the optimal portfolio composition.

3.3.3 Dependence of the optimal portfolio on liabilities

We now investigate the role of stochastic liabilities in the model with an already fixed scenario tree for assets. We start by computing VSS with respect to liabilities and showing different optimal portfolio compositions under variants R and S. Here VSS is computed as $(RP - EEV)/RP * 100$, where RP is the optimal objective value of the problem with stochastic liabilities, EEV is the objective value of the problem with stochastic liabilities evaluated at the optimal solution of the problem based solely on the expected value scenario for liabilities.

a	Variant	VSS (%)	Liabilities	Portfolio			
				(after rebalancing, market values %)			
				Dep	B1	B2	B3
0.4	R	4.26936E-05	EV	62	2	27	9
			Stoch	62	2	27	9
	S	0	EV	37	27	27	9
			Stoch	37	27	27	9
4	R	0.000357221	EV	90	0	1	9
			Stoch	90	0	1	9
	S	0	EV	37	27	27	9
			Stoch	37	27	27	9
				Portfolio (initial, market values %)			
				37	27	27	9

Table 12: VSS with respect to liabilities, all variants assuming NL

We have gained very little when allowing for stochastic liabilities, which we attribute to the low level and low variability of contributions, see Tables 5 and 13. Low level and low variability of contributions also yields low variability of λ which does not cause then extra penalties in the objective. The optimal objective value and the decision variables for the first period remain almost unchanged when shifting towards the expected value scenario. This is advantageous from the point of the view of numerical computations and will allow to solve problems with a considerably higher number of scenarios for assets. Moreover working with the expected values of cash flows on the liabilities side is often used in practice. Our result supports this simplified procedure, which in general leads to over-optimistic conclusions about the fund performance; cf. Subsection 2.2.1. It was obtained under rather restrictive assumptions: the accepted independence of the stochastic factors in the assets and liabilities tree, reduction of the number of scenarios for stochastic liabilities to 5 scenarios obtained by the scenario reduction algorithm and the planning horizon covering only three years (three stages).

We now check how sensitive are our results with respect to changes in the behavior of participants, i.e., under different assumptions about newly incoming and a changed propensity to the lump sum settlement. As an example, assume no newly incoming during the whole planning horizon, the propensity to the lump sum settlement increased by twenty percent and the propensity to a terminal settlement quadrupled. These assumptions are incorporated into the simulation model and the scenario generation continues as in 2.1.2. Table 13 gives scenarios of liabilities for this “no incoming” variant.

Scenario	*Period*			*Probability*
	F			
	1	2	3	
1	-2.165e7	4.312e7	-1.674e8	0.109
2	-8.004e7	4.615e7	-1.047e8	0.160
3	-8.004e7	-2.205e7	-1.388e8	0.273
4	-1.167e8	5.409e7	-1.982e8	0.102
5	-7.568e7	3.547e7	-1.500e8	0.355
	λ			
1	6.459e7	7.535e7	7.405e7	
2	6.281e7	7.357e7	7.411e7	
3	6.281e7	7.140e7	7.061e7	
4	6.175e7	7.267e7	6.989e7	
5	6.297e7	7.338e7	7.247e7	
	EF			
1	-7.583e7	2.418e7	-1.465e8	1
	$\sigma(F)/EF$			
	0.294	1.197	0.170	
	$E\lambda$			
1	6.295e7	7.301e7	7.214e7	1
	$\sigma(\lambda)/E\lambda$			
	0.011	0.016	0.021	

Table 13: Scenario tree for liabilities, after reduction, variant "no incoming"

Again we compare the results for different cases.

a	Variant	Liabilities	Portfolio (after rebalancing, market values %)			
			Dep	B1	B2	B3
0.4	R	normal	62	2	27	9
		noincoming	58	6	27	9
	S	normal	37	27	27	9
		noincoming	37	27	27	9
4	R	normal	90	0	1	9
		noincoming	79	0	12	9
	S	normal	37	27	27	9
		noincoming	60	27	4	9
			Portfolio (initial, market values %)			
			37	27	27	9

Table 14: Optimal portfolios at the beginning of the first period

Table 14 shows that the optimal solution of the ALM problem changes as a consequence of different specifications of inputs for the liabilities tree. A separation of asset management and liabilities management will not be optimal in our case. There is a little justification for working with stochastic liabilities; see Table 12.

3.3.4 Suggestions for future research

The model—a scenario-based multistage stochastic program with random recourse and linear constraints—does not fully capture the complexity of the real problem. For example

• specific taxation rules for pension funds could be described by auxiliary 0-1 (logic) variables;

• profit sharing can be calculated only after audit; hence, in the reality, there are random flows whose *probability distribution depends on decision variables*; as a simplification fixed valorization of accumulated wealth was used similarly as in [29];

• more complex risk considerations and multiple criteria could be applied, integer decision variables would reflect better the trading customs, accounting standards, etc.;

• inclusion of other permitted asset classes would require to model their return scenarios, taking into account their correlation with returns of the bond indices;

• it will be useful to extend the techniques designed for validation of results to cover a sensitivity analysis with respect to the moment values used in the moment fitting method described in 2.1.1 or to the fixed values of valorizations, to analyze the impact of the chosen branching scheme and the stress testing on the rebalancing strategy, etc.;

• a theoretically justified stability analysis of the optimal *solutions* for the ALM problem is highly needed.

4 Summary and conclusions

We developed an ALM model for defined contribution pension plans. The model distinguishes between cash flows and the accounting profit and it models quantities which are highly relevant for the fund manager. Both the market value and historical costs are tracked so that sensitivity of the optimal solution on the initial portfolio composition can be assessed. Portfolios with the same weights at the beginning of the planning horizon lead to different optimal solutions of the ALM problem when a different level of provisions is admitted.

Scenario generation procedures were selected regarding differences in the available data on assets returns and on liability flows and extended to take care for no-arbitrage requirements.

We analyzed the stability of the optimal value and of the optimal asset allocation with respect to changes in the portfolio of insured and in the assumed development of the market. The optimal solutions in our implementation of the ALM problem were insensitive to stochasticity embedded in the liability tree. Hence, it is possible to use only the expected value scenario for liabilities instead of the reduced scenario tree. Contamination bounds were applied to quantify the influence of including out-of-sample or stress scenarios on the optimal value.

It is not possible to separate the asset management and liabilities management problems. Changes in the parameters of the liabilities distribution, even a changed expected value, cause significant changes in the optimal solution, i.e., in the optimal portfolio composition of the ALM problem.

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